

# Standard Polynomial Equations over Division Algebras

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## Abstract

Given a central division algebra  $D$  of degree  $d$  over a field  $F$ , we associate to any standard polynomial  $\phi(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_0$  over  $D$  a “companion polynomial”  $\Phi(z)$  of degree  $nd$  with coefficients in  $F$ . The roots of  $\Phi(z)$  in  $D$  are exactly the set of conjugacy classes of the roots of  $\phi(z)$ . When  $D$  is a quaternion algebra, we explain how all the roots of  $\phi(z)$  can be recovered from the roots of  $\Phi(z)$ . Along the way, we are able to generalize a few known facts from  $\mathbb{H}$  to any division algebra. The first is the connection between the right eigenvalues of a matrix and the roots of its characteristic polynomial. The second is the connection between the roots of a standard polynomial and left eigenvalues of the companion matrix.

*Keywords:* Polynomial Equations, Division Algebras, Division Rings, Right Eigenvalues, Left Eigenvalues

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## 1. Introduction

Let  $F$  be a field. A *central division algebra* is a division algebra  $D$  which is of finite dimension over its center  $F$ . The dimension  $[D : F]$  is a square, and the integer  $d = \sqrt{[D : F]}$  is the *degree* of  $D$  over  $F$ .

Throughout this paper, we consider *standard polynomials* over  $D$ . These are monic polynomials with coefficients appearing only on the left-hand side of the variable:

$$\phi(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_0$$

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where  $c_i \in D$ . By a *root* of a standard polynomial  $\phi$  over  $D$  we mean an element  $\lambda \in D$  satisfying  $\phi(\lambda) = 0$ . We are interested in finding these roots.

One of the earliest papers to explore standard polynomials is [Niv41]. Niven proves that when  $D$  is a real quaternion algebra and  $F$  is a real closed field, every standard polynomial has a root in  $D$ . A good survey on polynomial equations over more general division rings can be found in [GM65].

The known results when it comes to polynomial equations over quaternion algebras can be outlined as follows: In [HS02] a formula was given for the roots of any quadratic (i.e.  $n = 2$ ) standard polynomial over  $\mathbb{H}$ . In [Abr09] this formula was generalized to any quaternion algebra over fields of characteristic not 2, and in [Cha15] to quaternion algebras over fields of characteristic 2 as well. In [SPV01] it was shown that the roots of any standard polynomial of degree  $n$  over  $\mathbb{H}$  are also roots of the “companion polynomial”. This polynomial is the characteristic polynomial of the companion matrix, which is a polynomial of degree  $2n$  with coefficients in  $\mathbb{R}$ . In [JO10] it was shown how the roots of the original polynomial can be recovered from the roots of the companion polynomial.

In this paper we generalize the result from [SPV01] to any central division algebra. In the special case of quaternion algebras, it provides a method for finding all the roots of a given standard polynomial. We make use of the theory of left and right eigenvalues, which was studied extensively for matrices over  $\mathbb{H}$  and here we present some generalizations to any central division algebra.

## 2. Right Eigenvalues of Matrices over Central Division Algebras

Let  $D$  be a central division algebra of degree  $d$  over  $F$  and let  $n$  be a positive integer. We write  $M_n(D)$  for the ring of  $n \times n$  matrices over  $D$ , and  $D^n$  for the column vector space of dimension  $n$  over  $D$ . The vector space  $D^n$  is both a left  $M_n(D)$ -module and a  $D$  bi-module.

A right eigenvalue of a matrix  $A$  in  $M_n(D)$  is an element  $\lambda \in D$  for which there exists a vector  $v \in D^n$  satisfying  $Av = v\lambda$ . A left eigenvalue is defined in a similar way, just with  $Av = \lambda v$ .

The ring  $M_n(D)$  can be identified with the ring of endomorphisms of  $D^n$  as a right  $D$ -module. Let  $K$  be a maximal subfield of  $D$ . The ring  $D$  can be viewed as a vector space over  $K$ , and in particular  $[D : K] = d$ . Therefore,  $D^n$  (as a right  $D$ -module) can be identified with the vector space  $K^{nd}$ . Let  $g : D^n \rightarrow K^{nd}$  be a  $K$ -vector space isomorphism. Note that  $g(v\lambda) = g(v)\lambda$

for any  $v \in D^n$  and  $\lambda \in K$ . Via this identification, each element in  $M_n(D)$  can be identified with a  $K$ -linear endomorphism of  $K^{nd}$ . This gives rise to a  $K$ -linear map  $f : M_n(D) \rightarrow M_{nd}(K)$ , which is an embedding. Note that for any  $A \in M_n(D)$  and  $v \in D^n$ , we have  $g(Av) = f(A)g(v)$ . This can be expressed in the following commutative diagram:

$$\begin{array}{ccc} D^n & \xrightarrow{g} & K^{nd} \\ \downarrow \cdot A & & \downarrow \cdot f(A) \\ D^n & \xrightarrow{g} & K^{nd} \end{array}$$

For each  $A \in M_n(D)$ , the *characteristic polynomial* of  $A$  is

$$\Phi_A(z) := \det(f(A) - zI)$$

where  $I$  is the identity matrix in  $M_{nd}(K)$ . The coefficients of  $\Phi_A(z)$  are known to lie in  $F$ . Furthermore, they are independent of both the choice of maximal subfield  $K$  of  $D$  as well as choice of isomorphism  $g : D^n \rightarrow K^{nd}$  (see for example [GS06, Section 4.5]).

We say that two elements  $d_1$  and  $d_2$  in  $D$  are *conjugate* if there exists some nonzero  $q \in D$  such that  $d_1 = qd_2q^{-1}$ . The conjugacy class of an element in  $D$  is the set of all its conjugates.

It was shown in [Lee49] that the conjugacy classes of right eigenvalues of an  $n \times n$  matrix  $A$  over  $\mathbb{H} = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + ij\mathbb{R}$  are the conjugacy classes of roots of the complex roots of the characteristic polynomial of  $\begin{pmatrix} A_1 & -\overline{A_2} \\ A_2 & \overline{A_1} \end{pmatrix}$  where  $A_1$  and  $A_2$  are the  $n \times n$  matrices over  $\mathbb{R}(i)(\cong \mathbb{C})$  satisfying  $A = A_1 + jA_2$ . We now generalize this fact to any central division algebra.

**Theorem 2.1.** *Let  $D$  be a central division  $F$ -algebra of degree  $d$ , and  $n$  be a positive integer. Let  $A \in M_n(D)$  and  $\lambda \in D$ . Then  $\lambda$  is a right eigenvalue of  $A$  if and only if  $\Phi_A(\lambda) = 0$ .*

*Proof.* Let  $K$  be a maximal subfield of  $D$  containing  $\lambda$ . Let  $g : D^n \rightarrow K^{nd}$  and  $f : M_n(D) \rightarrow M_{nd}(K)$  be as defined above.

Assume  $\lambda$  is a right eigenvalue of  $A$ . Then there exists a nonzero vector  $v \in D^n$  such that  $Av = v\lambda$ . Then  $f(A)g(v) = g(Av) = g(v\lambda) = g(v)\lambda$ . The element  $\lambda$  is an eigenvalue of  $f(A)$  (in the classical sense), and so a root of  $\Phi_A(z)$ .

Assume  $\Phi_A(\lambda) = 0$ . Then  $\lambda$  is an eigenvalue of  $f(A)$ , so there exists a nonzero vector  $w \in K^{nd}$  such that  $f(A)w = w\lambda$ . Let  $v = g^{-1}(w)$ . Then  $Av = v\lambda$ , hence  $\lambda$  is a right eigenvalue of  $A$ .  $\square$

Note that since  $\Phi_A(z)$  has coefficients in  $F$ , for each root  $\lambda \in D$  of  $\Phi_A(z)$ , all of its conjugates are roots as well. So one can consider the roots of  $\Phi_A(z)$  as a collection of conjugacy classes of elements of  $D$ . Each such conjugacy class corresponds to the isomorphism class of a finite field extension  $F(\lambda)/F$ , which can be identified with a subfield of a fixed algebraic closure  $\overline{F}$  of  $F$ . Therefore, in order to find all the roots of  $\Phi_A(z)$  that lie in  $D$ , one can solve it as a polynomial over  $\overline{F}$ , and then for each root  $\lambda \in \overline{F}$  check whether  $F(\lambda)$  is a subfield of  $D$ . If so,  $\lambda$  is a root of  $\Phi_A(z)$  in  $D$ , and otherwise it is not.

### 3. Standard Polynomials and Left Eigenvalues of the Companion Matrix

Let  $\phi(z) = z^n + c_{n-1}z^{n-1} + \dots + c_0$  be a standard polynomial with coefficients  $c_0, \dots, c_{n-1}$  in  $D$ . We want to find the roots of  $\phi(z)$  in  $D$ , i.e. all the elements  $\lambda \in D$  satisfying  $\phi(\lambda) = 0$ .

Given such a polynomial, we define its companion matrix to be

$$C_\phi = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \dots & 0 & 1 \\ -c_0 & -c_1 & \dots & -c_{n-2} & -c_{n-1} \end{pmatrix}.$$

**Theorem 3.1.** *The roots of  $\phi(z)$  are exactly the left eigenvalues of  $C_\phi$ . Furthermore, given such a left eigenvalue  $\lambda$  of the companion matrix  $C_\phi$ , the*

*vector  $v = \begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{pmatrix}$  is a corresponding left eigenvector, i.e. a vector satisfying  $C_\phi v = \lambda v$ .*

*Proof.* The element  $\lambda \in D$  is a left eigenvalue of  $C_\phi$  if and only if there exists a nonzero vector

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in D^n$$

satisfying  $C_\phi v = \lambda v$ . This equality is equivalent to the system

$$\begin{aligned} v_2 &= \lambda v_1 \\ \vdots & \\ v_n &= \lambda v_{n-1} \\ -c_0 v_1 - \cdots - c_{n-1} v_n &= \lambda v_n \end{aligned}$$

Note that since  $v \neq 0$ ,  $v_1 \neq 0$ . The first  $n - 1$  equations mean that  $v$  is

$$\begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{pmatrix} v_1 \text{ and the last equation then becomes}$$

$$(c_0 + c_1 \lambda + \cdots + c_{n-1} \lambda^{n-1} + \lambda^n) v_1 = 0.$$

By dividing by  $v_1$  from the right, we obtain that  $\lambda$  is a root of  $\phi(z)$ .

In the other direction, it is straight-forward to see that for any  $\lambda \in D$  satisfying  $c_0 + c_1 \lambda + \cdots + c_{n-1} \lambda^{n-1} + \lambda^n = 0$ ,

$$C_\phi \begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{pmatrix}$$

which means that  $\lambda$  is a left eigenvalue of  $C_\phi$ . □

**Remark 3.2.** *Similar connections between the polynomial and its companion matrix were also pointed out in [LLO08]. The fact that every root of  $\phi(z)$  is also a left eigenvalue of  $C_\phi$  can also be obtained as a result of Lemma 4.7 in that paper.*

**Corollary 3.3.** *Every left eigenvalue of  $C_\phi$  is also a right eigenvalue.*

*Proof.* Let  $\lambda$  be a left eigenvalue. Then  $\begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{pmatrix}$  is the corresponding

eigenvector. Now

$$\begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{pmatrix} \lambda = \lambda \begin{pmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{pmatrix}.$$

□

**Corollary 3.4.** *Every right eigenvalue of  $C_\phi$  is conjugate to some left eigenvalue of  $C_\phi$ .*

*Proof.* Assume  $\lambda$  is a right eigenvalue. Then there exists a nonzero vector

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in D^n$$

satisfying  $C_\phi v = v\lambda$ . From this equality we obtain the system

$$\begin{aligned} v_2 &= v_1 \lambda \\ &\vdots \\ v_n &= v_{n-1} \lambda \\ -c_0 v_1 - \cdots - c_{n-1} v_n &= v_n \lambda \end{aligned}$$

Substituting the first  $n - 1$  equations in the last one we obtain

$$c_0 v_1 + c_1 v_1 \lambda + \cdots + c_{n-1} v_1 \lambda^{n-1} + v_1 \lambda^n = 0.$$

Recall that since  $v$  is nonzero,  $v_1$  is nonzero. Take  $\lambda' = v_1 \lambda v_1^{-1}$ . Then  $\lambda'$  is a root of  $\phi(z)$  and so a left eigenvalue of  $C_\phi$  □

Let the characteristic polynomial of  $C_\phi$  be denoted by  $\Phi(z)$ . We call it the *companion polynomial* of  $\phi(z)$ . Note that it is of degree  $nd$  and has coefficients in  $F$ .

**Remark 3.5.** *For a quaternionic (i.e. degree 2) division algebra, it is not mentioned explicitly in [JO10], but the companion polynomial  $q(z) = z^{2n} + (\overline{c_{n-1}} + c_{n-1})z^{2n-1} + \dots$  associated to any polynomial  $p(z) = z^n + c_{n-1}z^{n-1} + \dots$  is indeed the characteristic polynomial of the companion matrix. This can be easily verified by a straight-forward computation: take the companion matrix*

$C_p$ , write it as  $A_1 + jA_2$  for the appropriate  $A_1, A_2 \in M_n(\mathbb{R}(i))$  and compute the characteristic polynomial of

$$\begin{pmatrix} A_1 & -\overline{A_2} \\ A_2 & \overline{A_1} \end{pmatrix}.$$

**Theorem 3.6.** *The roots of  $\phi(z)$  are also roots of  $\Phi(z)$ , and each conjugacy class of roots of  $\Phi(z)$  contains a root of  $\phi(z)$ .*

*Proof.* By Theorem 3.1 the roots of  $\phi(z)$  are left eigenvalues of the companion matrix. By Corollary 3.3 those left eigenvalues are also right eigenvalues, and by Theorem 2.1 the right eigenvalues are roots of the companion polynomial. The second assertion is immediate from Corollary 3.4.  $\square$

#### 4. Standard Polynomials over Quaternion Algebras

Let  $\phi(z)$  be again a standard polynomial over a central division algebra  $D$  of degree  $d$  over  $F$ . The algebra  $D$  is called a *quaternion algebra* if  $d = 2$ . We begin this section with a general discussion, and we conclude by explaining how the roots of  $\phi(z)$  can be recovered from the roots of its companion polynomial  $\Phi(z)$  in the case that  $D$  is a quaternion algebra.

Every element  $\lambda \in D$  has a characteristic polynomial, which means that  $\lambda^d + b_{d-1}\lambda^{d-1} + \cdots + b_0 = 0$  for some  $b_{d-1}, \dots, b_0 \in F$ . Note that the characteristic polynomial of  $\lambda$  does not change if we conjugate  $\lambda$  by any  $q \in D^\times$ . Write  $C_{d-1}(z), \dots, C_0(z)$  for the functions from  $D$  to  $F$  such that  $z^d + C_{d-1}(z)z^{d-1} + \cdots + C_0(z)$  is the characteristic polynomial of  $z$ . We may write  $\phi(z)$  as  $\psi_{d-1}(z)z^{d-1} + \cdots + \psi_0(z)$  using the functions  $C_{d-1}(z), \dots, C_0(z)$ .

**Example 4.1.** *If  $\phi(z) = z^3$  and  $d = 2$  then*

$$\begin{aligned} \phi(z) &= z^2 \cdot z + c = (-C_1(z)z - C_0(z))z + c = -C_1(z)z^2 - C_0(z)z \\ &= -C_1(z)(-C_1(z)z - C_0(z)) - C_0(z)z \\ &= (C_1(z)^2 - C_0(z))z + C_1(z)C_0(z) \end{aligned}$$

and so  $\psi_1(z) = C_1(z)^2 - C_0(z)$  and  $\psi_0(z) = C_1(z)C_0(z)$ .

By Theorem 3.6, the roots of  $\phi(z)$  are also roots of  $\Phi(z)$ . Assuming we have all the roots of  $\Phi(z)$  on hand, we want to recover the roots of  $\phi(z)$ . The following statement is immediate:

**Proposition 4.2.** *Given a root  $\lambda$  of  $\Phi(z)$ ,  $\lambda$  is a root of  $\phi(z)$  if and only if  $\lambda$  is a root of the polynomial  $\Psi(z) = \psi_{d-1}(\lambda)z^{d-1} + \cdots + \psi_0(\lambda)$ .*

In the case  $d = 2$ , the polynomial  $\Psi(z)$  defined in the proposition above is either a linear polynomial or a constant. It cannot be a nonzero constant, because that would mean that no element in the conjugacy class of  $\lambda$  is a root of  $\phi$ , contradictory to Theorem 3.6. We now have a way of finding roots of  $\phi(z)$  from the roots of its companion polynomial  $\Phi(z)$  when  $D$  is a quaternion algebra:

**Corollary 4.3.** *Assume  $d = 2$  and fix a root  $\lambda$  of  $\Phi(z)$ . If  $\Psi(z)$  is constantly zero then every element in the conjugacy class of  $\lambda$  is a root of  $\phi(z)$ . If  $\Psi(z)$  is a linear polynomial, then the root of  $\phi(z)$  in the conjugacy class of  $\lambda$  can be found simply by solving the linear equation  $\Psi(z) = 0$ .*

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